LOWNESS OF HIGHER RANDOMNESS NOTIONS

BY

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ABSTRACT

We study randomness notions given by higher recursion theory, estab-
lishing the relationships Π_1^1-randomness $\subset$ Π_1^1-Martin-Löf randomness $\subset$
Δ_1^1-randomness $=$ Δ_1^1-Martin-Löf randomness. We characterize the set
of reals that are low for Δ_1^1 randomness as precisely those that are Δ_1^1
-traceable. We prove that there is a perfect set of such reals.

1. Introduction

In recent years the study of algorithmic randomness has been focused almost exclusively on the arithmetical hierarchy, and with considerable success. In particular, n-randomness and weak n-randomness were investigated for $n < \omega$. (Recall here that a real number in 2^ω is n-random if it is not in the intersection of any nested uniformly Σ_n^0 sequence $(V_n)_{n\in\omega}$ of sets of reals so that $\mu(V_n) \leq 2^{-n}$. A real number is weakly n-random if it is not in any Π_n^0 null set of reals.) Nevertheless, the conceptualization of algorithmic randomness may be approached from a different direction. If one accepts the view that a real number is random if it does not satisfy any "reasonable" collection of properties of measure zero, then it makes sense to study randomness relative to a naturally defined notion, and investigate the mathematical properties of reals that are random in the given context. There are two ways of doing this: The first is to study algorithmic n-randomness by varying the notion of the underlying measure (recent work of Reimann and Slaman, to appear, points to a significant link between being n-random and the measure that determines randomness), while the second is to retain the classical notion of Lebesgue measure and raise the logical complexity of the sets of reals being considered in the investigation of randomness. In this paper we adopt the second approach and consider randomness within the realm of second order arithmetic. In the spirit of higher recursion theory, we call this the theory of higher randomness.

From the point of view of higher recursion theory, a natural extension of the notion of recursive enumerability for subsets of ω in second order arithmetic is Π_1^1 definability. An extensive theory has been developed by Kleene, Spector, Gandy, Sacks and others (cf. Sacks [18] for a thorough treatment of the subject). Martin-Löf [11] was the first to study randomness in the setting of higher recursion theory, when he showed that the intersection of a sequence of hyperarithmetical sets of reals of measure one forms a nonempty Σ_1^1 set. For almost 40 years this remained the only contribution to the subject of higher randomness, with the marginal exception of Sacks [18] (Chapter IV, Exercise 2.5). He defined what we call in this paper Π_1^1 and Δ_1^1 random reals, namely those reals avoiding Π_1^1 and Δ_1^1 null sets, respectively. The recent work of Hjorth and Nies [5] may be regarded as the first systematic study of randomness via effective descriptive set theory. In this paper we follow the same direction, by examining

various notions we consider to be central to any reasonable theory of randomness. We study them in the setting of higher recursion theory. The motivation is to understand how the choice of a mathematical definability setting determines the key properties of random reals within the structure. We first investigate the analogs of various naturally defined, competing and inequivalent notions of randomness in first order theory. We show that under some circumstances, their analogs are equivalent in second order arithmetic. For instance, a real number x is Δ_1^1 random if and only if it is Δ_1^1 random in the sense of Martin-Löf tests. In the case when $\omega_1^x = \omega_1^{\mathrm{CK}}$, the equivalence extends to x being Π_1^1 random and being Π_1^1 random in the sense of Martin-Löf (Theorem 3.3 and Corollary 3.5; see §3 for the definitions). In general, however, the last two notions do not coincide (Theorem 3.12). In §4 we study an analog of the notion of a real number of hyperimmune-free degree, being Δ_1^1 dominated. We show that the set of Δ_1^1-dominated reals has measure 1, and that every Π_1^1-random real number is Δ_1^1-dominated (Theorem 4.2 and Corollary 4.3). In §5 we study the class of Δ_1^1 and Π_1^1-traceable sets as analogs of recursive and r.e. traceable reals. It turns out that these two classes are identical (of size the continuum, Theorem 5.4), and properly contained in the class of Δ_1^1 dominated reals. This is used to study the class of low for Δ_1^1 random reals where it is proved in §6 Theorem 6.2 that a real number is low for Δ_1^1 random if and only if it is Δ_1^1-traceable. We end the paper with further comments on higher randomness, one result on low for Π_1^1-randomness, and some open problems.

2. Preliminaries

We assume that the reader is familiar with elements of higher recursion theory, as presented, for instance, in Sacks [18]. Fix a standard Π_2^0 set $H \subseteq \omega \times 2^\omega \times 2^\omega$ so that for all x and $n \in \mathcal{O}$, there is a unique real number y satisfying $H(n, x, y)$. Moreover, if $\omega_1^x = \omega_1^{\mathrm{CK}}$, then each real number $z \leq_h x$ is Turing reducible to some y so that $H(n, x, y)$ holds for some $n \in \mathcal{O}$. Roughly speaking, y is the $|n|$-th Turing jump of x. These y's are called H^x sets and denoted by H_n^x's.

We use the Cantor pairing function, the bijection $p : \omega^2 \to \omega$ given by $p(n, s) = \frac{(n+s)^2 + 3n + s}{2}$, and write $\langle n, s \rangle = p(n, s)$.

The following results will be used in later sections.

THEOREM 2.1 (Gandy): *If $A \subset 2^\omega$ is a nonempty Σ_1^1 set, then there is a real number $x \in A$ so that $\mathcal{O}^x \leq_h \mathcal{O}$.*

THEOREM 2.2 (Spector [19] and Gandy [4]): *$A \subset 2^\omega$ is Π_1^1 if and only if there is an arithmetical predicate $P(x,y)$ such that*

$$y \in A \leftrightarrow \exists x \leq_h y P(x,y).$$

THEOREM 2.3 (Sacks[17]): *If x is non-hyperarithmetical, then*

$$\mu(\{y : y \geq_h x\}) = 0.$$

THEOREM 2.4 (Sacks [18]): *The set $\{x : x \geq_h \mathcal{O}\}$ is Π_1^1. Moreover, $x \geq_h \mathcal{O}$ if and only if $\omega_1^x > \omega_1^{\mathrm{CK}}$.*

A consequence of the last two theorems above is that the set $\{x : \omega_1^x > \omega_1^{\mathrm{CK}}\}$ is a Π_1^1 null set.

The ramified analytical hierarchy was introduced by Kleene, and applied by Fefferman [2] and Cohen [1] to study forcing, a tool that turns out to be powerful in the investigation of higher randomness theory. We recall some basic facts here Sacks [18], whose notation we mostly follow:

The ramified analytic hierarchy language $\mathfrak{L}(\omega_1^{\mathrm{CK}}, \dot{x})$ contains the following symbols:

(1) Number variables: $j, k, m, n, \ldots$;
(2) Numerals: 0,1,2,…;
(3) Constant: $\dot{x}$;
(4) Ranked set variables: $x^\alpha, y^\alpha, \ldots$ where $\alpha < \omega_1^{\mathrm{CK}}$;
(5) Unranked set variables: $x, y, \ldots$;
(6) Others symbols include: $+, \cdot$ (times), $'$ (successor) and $\in$.

Formulas are built in the usual way. A formula φ is **ranked** if all of its set variables are ranked. Due to its complexity, the language is not codable in a recursive set but rather in the countable admissible set $L_{\omega_1^{\mathrm{CK}}}$.

To code the language in a uniform way, we fix a Π_1^1 path $\mathcal{O}_1$ through $\mathcal{O}$ (by [3] such a path exists). Then a ranked set variable x^α is coded by the number $(2, n)$ where $n \in \mathcal{O}_1$ and $|n| = \alpha$. Other symbols and formulas are coded recursively. With such a coding, the set of Gödel number of formulas is Π_1^1. Moreover, the set of Gödel numbers of ranked formulas of rank less than α is r.e. uniformly in the unique notation for α in $\mathcal{O}_1$. Hence, there is a recursive function f so that

$W_{f(n)}$ is the set of Gödel numbers of the ranked formula of rank less than $|n|$ when $n \in \mathcal{O}_1$ ($\{W_e\}_e$ is, as usual, an effective enumeration of r.e. sets).

One now defines a structure $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$, where x is a real number, analogous to the way Gödel's L is defined, by induction on the recursive ordinals. Only at successor stages are new sets defined in the structure. The reals constructed at a successor stage are arithmetically definable by the reals constructed at earlier stages. The details may be found in [18]. We define $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi$ for a formula φ of $\mathfrak{L}(\omega_1^{\mathrm{CK}}, \dot{x})$ by allowing the unranked set variables to range over $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$, while the symbol x^α will be interpreted as the reals built before stage α. In fact, the domain of $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$ is the set $\{y : y \leq_h x\}$ if and only if $\omega_1^x = \omega_1^{\mathrm{CK}}$ (see [18]).

A sentence φ of $\mathfrak{L}(\omega_1^{\mathrm{CK}}, \dot{x})$ is said to be Σ_1^1 if it is ranked, or of the form $\exists x_1, \ldots, \exists x_n \psi$ for some formula ψ with no unranked set variables bounded by a quantifier.

We have the following result which is a model-theoretic version of the Gandy-Spector Theorem.

THEOREM 2.5 (Sacks [18]): *The set $\{(n_\varphi, x) : \varphi \in \Sigma_1^1 \wedge \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi\}$ is Π_1^1, where n_φ is the Gödel number of φ. Moreover, for each Π_1^1 set $A \subseteq 2^\omega$, there is a formula $\varphi \in \Sigma_1^1$ so that*

(1) $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi \implies x \in A;$
(2) *if $\omega_1^x = \omega_1^{\mathrm{CK}}$, then $x \in A \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi$.*

Note that if φ is ranked, then both the sets $\{x : \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi\}$ (the Gödel number of φ is omitted) and $\{x : \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \neg\varphi\}$ are Π_1^1 and so Δ_1^1. Moreover, if $A \subseteq 2^\omega$ is Δ_1^1, then there is a ranked formula φ so that $x \in A \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi$ (see Sacks [18]).

THEOREM 2.6 (Sacks [17]): *The set $\{(n_\varphi, p) : \mu(\{x | \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi\}) > p \wedge \varphi \in \Sigma_1^1 \wedge p$ is a rational number$\}$ is Π_1^1 where n_φ is the Gödel number of φ.*

THEOREM 2.7 (Sacks [17]): *There is a recursive function $f : \omega \times \mathbb{Q} \to \omega$ so that for all n which is Gödel number of a ranked formula*

(1) *$f(n, p)$ is Gödel number of a ranked formula;*
(2) *The set $\{x : \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_{f(n,p)}\} \supseteq \{x : \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_n\}$ is open;*
(3) *$\mu(\{x : \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_{f(n,p)}\} - \{x : \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_n\}) < p$.*

THEOREM 2.8 (Sacks [17] and Tanaka [21]): *If A is a Π_1^1 set of positive measure, then A contains a hyperarithmetical real.*

3. Defining higher randomness notions

A sequence of open sets $\{U_n\}_{n\in\omega}$ is a Martin-Löf test (ML-test) if $\mu(U_n) \leq 2^{-n}$ for all n. Given a class of sets of reals $\mathbf{\Gamma}$ (e.g. Π_1^1 or Δ_1^1), $\{U_n\}_{n\in\omega}$ is a $\mathbf{\Gamma}$-ML test if $\{(n,\sigma) : \sigma \in 2^{<\omega} \wedge [\sigma] \in U_n\} \in \mathbf{\Gamma}$.

Definition 3.1: Given a class $\mathbf{\Gamma}$ of sets of reals,

   (1) A real number x is $\mathbf{\Gamma}$-random if no $\mathbf{\Gamma}$ null set contains x.
   (2) A real number x is $\mathbf{\Gamma}$-ML-random if $x \notin \bigcap_{n\in\omega} U_n$ for any Γ ML-test $\{U_n\}_n$.

In this paper, we focus on Δ_1^1-ML, Δ_1^1-, Π_1^1-ML and Π_1^1-randomness. First we show that Δ_1^1-randomness and Δ_1^1-ML-randomness coincide. For this we need a lemma which will also be used later on. It says that at the hyperarithmetical level, the analogs of computable randomness and Schnorr randomness are the same.

LEMMA 3.2: *Let $\mathcal{A}$ be a null Δ_1^1 set. Then $\mathcal{A} \subseteq \bigcap U_n$ for some Δ_1^1-ML test $\{U_n\}_{n\in\omega}$ such that, in addition, $\mu(U_n) = 2^{-n}$ for each n.*

Proof. If $\mathcal{A}$ is a Δ_1^1-null set, then by Theorem 2.7 there is a recursive sequence of Δ_1^1 open sets U_n for which $\mu(U_n) < 2^{-n}$ and $\mathcal{A} \subseteq U_n$ for all n. So $\{U_n\}_{n\in\omega}$ is a Δ_1^1-ML-test.

It now suffices to show that the Δ_1^1-ML test $\{U_n\}_{n\in\omega}$ can be improved to a Δ_1^1-ML test $\{\hat{U}_n\}_{n\in\omega}$ such that $U_n \subseteq \hat{U}_n$ and $\mu(\hat{U}_n) = 2^{-n}$ for each n. For this, it clearly suffices to show that for each Δ_1^1 open set $S \subseteq 2^\omega$ and each rational $q \geq \mu(S)$ one can effectively obtain a Δ_1^1 open set $\hat{S}$ such that $S \subseteq \hat{S} \subseteq 2^\omega$ and $\mu(\hat{S}) = q$. Recall the isometry F between the conull subset of Cantor space 2^ω consisting of the coinfinite sets and the interval $I = [0,1)_{\mathbb{R}}$: for a coinfinite set $z \subseteq \omega$, let

$$F(z) = \sum_{i\in z} 2^{-i-1}.$$

Note that under F, the product measure μ turns into Lebesgue measure, and the lexicographical ordering $<_L$ becomes the usual ordering of real number numbers.

The function $f : I \to I$ given by $f(x) = \mu([0, x) \cup F(S))$ is continuous, non-decreasing and $f(0) \leq q$ while $f(x) \geq x$ for each $x \in I$. Thus there is a least r such that $f(r) = q$. Since $f \in \Delta_1^1$ and the left cut of r is $\{s \in \mathbb{Q} : f(s) < q\}$, the real number r is Δ_1^1, so $F(z) = r$ for some hyperarithmetical coinfinite $z \subseteq \omega$. Now the open set $\hat{S} = \{y : y <_L z\} \cup S$ is as desired.   ∎

THEOREM 3.3: *The following are equivalent for a real number x.*

   (i) x is Δ_1^1-random;
   (ii) x is Δ_1^1-ML-random.

Proof. (i) $\Rightarrow$ (ii): If $\{\hat{U}_n\}_{n \in \omega}$ is a Δ_1^1-ML-test, then $V = \bigcap_{n \in \omega} \hat{U}_n$ is a Δ_1^1 null set. So $x \notin V$.

(ii) $\Rightarrow$ (i): This is an immediate consequence of the previous lemma.   ∎

Hjorth and Nies [5] gave a direct proof of the result that the union of all Π_1^1 null sets is Π_1^1, which may also be obtained as a special case of the more general result [7, Theorem 1A-2]. We give yet another proof via the ramified analytical hierarchy, in order to extract more information about the set.

THEOREM 3.4 (Kechris [7]; Hjorth and Nies [5]): *The largest null Π_1^1 set exists.*

Proof. Define

$$\mathcal{P} = \{(n, x) : n \text{ is the Gödel number of a ranked formula}$$

$$\bigwedge \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \vDash \varphi_n(\dot{x}) \ \wedge \mu(\{x : \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \vDash \neg\varphi_n(\dot{x})\}) \geq 1)\}$$

and

$$\mathcal{Q}_n = \{x : (n, x) \in \mathcal{P}\}.$$

Define

$$\mathcal{Q} = \bigcup_{n \in \omega} \mathcal{Q}_n \cup \{x : \omega_1^x > \omega_1^{\mathrm{CK}}\}.$$

We show that $\mathcal{Q}$ is the largest null Π_1^1 set. By Theorem 2.6, the sequence $\{\mathcal{Q}_n\}_{n \in \omega}$ is a Π_1^1-sequence of Δ_1^1 sets. $\mathcal{Q}$ is Π_1^1. Moreover, $\mu(\mathcal{Q}_n) = 0$ for all $n \in \omega$. Since $\mu(\{x : \omega_1^{\mathrm{CK}} = \omega_1^x\}) = 1$, $\mu(\mathcal{Q}) = 0$.

If A is a Π_1^1 null set, then, by Theorem 2.5, there is a ranked formula $\varphi \in \Sigma_1^1$ so that if $\omega_1^x = \omega_1^{\mathrm{CK}}$, then $x \in A \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \exists y \varphi(\dot{x}, y)$. So if $\omega_1^x = \omega_1^{\mathrm{CK}}$, then $x \in A \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \exists y^\alpha \varphi(\dot{x}, y^\alpha)$ for some $\alpha < \omega_1^{\mathrm{CK}}$. Since the set $\{x : \omega_1^x > \omega_1^{\mathrm{CK}}\}$ is null, it is easy to see that $A \subseteq \mathcal{Q}$.   ∎

COROLLARY 3.5: *Suppose $\omega_1^x = \omega_1^{\mathrm{CK}}$. Then x is Δ_1^1-random if and only if x is Π_1^1-ML-random, and this is equivalent to x being Π_1^1-random.*

Proof. Clearly Π_1^1-randomness implies Π_1^1-ML-randomness. By Theorem 3.3, it suffices to show that if x is Δ_1^1-random and $\omega_1^x = \omega_1^{\mathrm{CK}}$, then x is Π_1^1-random. Assume $\omega_1^x = \omega_1^{\mathrm{CK}}$. If x is Δ_1^1-random, then $x \notin \mathcal{Q}_n$ for all n. So $x \notin \mathcal{Q}$. Hence x is Π_1^1-random. ∎

In contrast to Theorem 3.4, we have the following.

PROPOSITION 3.6: *There is no largest null Σ_1^1 set.*

Proof. Suppose A is the largest null Σ_1^1 set. Then by the Tanaka-Sacks Theorem 2.8, there is a Δ_1^1 real number $x \notin A$. $X = \{x\}$ is Δ_1^1 and $X \cap A = \emptyset$, a contradiction. ∎

By Theorem 2.4 and the proof of Theorem 3.4, we have the following result.

PROPOSITION 3.7 (Hjorth and Nies [5]): *If x is Π_1^1-random, then $\omega_1^x = \omega_1^{\mathrm{CK}}$.*

Together with Corollary 3.5, the Π_1^1-random reals are precisely the Δ_1^1-random reals x that also satisfy $\omega_1^x = \omega_1^{\mathrm{CK}}$.

By the Gandy Basis Theorem 2.1, there is a Π_1^1-random real number x with $\mathcal{O}^x \leq_h \mathcal{O}$.

THEOREM 3.8 (Hjorth and Nies [5]): *Given any real number x, there is a Π_1^1-ML-random real number $y \geq_h x$.*

Combining Theorem 3.8 and Proposition 3.7, we have the following consequence.

COROLLARY 3.9 (Hjorth and Nies [5]): *There is a Π_1^1-ML-random real number that is not Π_1^1-random.*

We now separate Δ_1^1-randomness from Π_1^1-ML-randomness, which is needed for the proof of Theorem 3.12 below. If one views the randomness notions as operators mapping oracles to classes, the separation can be obtained as a consequence of Theorem 5.4, Theorem 6.2, and the result of Hjorth and Nies [5] that every low for Π_1^1-ML-random real number is hyperarithmetical. We now obtain the separation for the plain randomness notions. Recall that in [5] a Π_1^1 version of prefix free Kolmogorov complexity was introduced, denoted

by K. It was shown that a Theorem analogous to the one of Schnorr holds, namely: z is Π^1_1-ML-random if and only if there is a $b \in \omega$ such that for each n, $K(z \upharpoonright n) \geq n - b$. So the following result implies the separation.

THEOREM 3.10: *Let h be a nondecreasing Δ^1_1 function such that*

$$\lim_n h(n) = \infty.$$

Then there is a Δ^1_1-random real number z such that $\forall^\infty n\ K(z \upharpoonright n \mid n) \leq h(n)$.

Here, $K(\sigma \mid n)$ is the complexity of σ given n. A number n is encoded in some effective way by a string (say, the binary expansion). Then $K(\sigma) \leq K(\sigma \mid n) + 2\log n$ (up to constants), so if we let $h(n) = \log n$ then we obtain $K(z \upharpoonright n) \leq 3\log n$.

First we need some preliminaries. A function

$$f : 2^{<\omega} \to \mathbb{R}^+ \cup \{0\}$$

is hyperarithmetical if there is a hyperarithmetical approximation function

$$g : 2^{<\omega} \times \omega \to \mathbb{Q}^+ \cup \{0\}$$

such that for each σ and n, we have $|f(\sigma) - g(\sigma, n)| \leq 2^{-n}$. A hyperarithmetical martingale is a hyperarithmetical function $M : 2^{<\omega} \to \mathbb{R}^+ \cup \{0\}$ that satisfies for every $\sigma \in 2^{<\omega}$ the martingale equality $M(\sigma 0) + M(\sigma 1) = 2M(\sigma)$. For a martingale M and a real number z, let $M(z) = \sup_n M(z \upharpoonright n)$. We say that the martingale M succeeds on z if the capital it reaches along z is unbounded, that is, $M(z) = \infty$. Let $S(M) = \{z : M \text{ succeeds on } z\}$.

Of course, $S(M)$ is a Δ^1_1 null set for any hyperarithmetical martingale M. Here is the converse. This equivalence is an effectivization of Ville's theorem.

LEMMA 3.11: *Let $\mathcal{A}$ be a Δ^1_1 null set. Then there is a hyperarithmetical martingale $M_\mathcal{A}$ such that $\mathcal{A} \subseteq S(M)$.*

Proof. By Lemma 3.2 there is a Δ^1_1 ML test $\{U_n\}_{n \in \omega}$ such that $\mu(U_n) = 2^{-n}$ and $\mathcal{A} \subseteq U_n$ for all n. Let $M_n(\sigma) = \mu(U_n \cap [\sigma])2^{|\sigma|}$. Then M_n is a hyperarithmetical martingale, uniformly in n, and $M_n(z) = 1$ if $z \in U_n$. Moreover, the start capital $M_n(\emptyset)$ is 2^{-n}. Now let $M(\sigma) = \sum_n M_n(\sigma)$, then M is as required. ∎

The proof of Theorem 3.10 is a straightforward computably random reals, see, for instance [14, Chap. 7]. We build a real number z of slowly growing initial

segment complexity (in the sense above) on which no $\mathbb{Q}$-valued hyperarithmetical martingale L succeeds. The martingale $M_{\mathcal{A}}$ is not necessarily $\mathbb{Q}$-valued, but by adaptation of a standard argument due to Schnorr (ibid.), for each hyperarithmetical martingale M there is a $\mathbb{Q}$-valued hyperarithmetical martingale $\hat{M}$ such that $\hat{M}(\sigma) \geq M(\sigma)$ for each σ.

In the following theorem we summarize the implications between the various randomness notions.

THEOREM 3.12:
$$\begin{aligned} \Delta_1^1(\mathcal{O})\text{-}randomness \quad &\Rightarrow \quad \Pi_1^1\text{-}randomness \\ &\Rightarrow \quad \Pi_1^1\text{-}ML\text{-}randomness \\ &\Rightarrow \quad \Delta_1^1\text{-}randomness \\ &\Leftrightarrow \quad \Delta_1^1\text{-}ML\text{-}\ randomness, \end{aligned}$$
and none of the implications may be reversed.

Proof. $\Delta_1^1(\mathcal{O})$-randomness $\Rightarrow \Pi_1^1$-randomness: Fix an $\mathcal{O}$-recursive well-ordering $<_R$ on ω of order type ω_1^{CK}. Then

$$\omega_1^x > \omega_1^{\mathrm{CK}} \Leftrightarrow \exists S \subseteq \omega \times \omega \, \exists f \in \omega^\omega$$
$$S \leq_T x \wedge \forall n \exists m(f(m) = n) \wedge \forall n \forall m(S(n,m) \Longleftrightarrow f(n) <_R f(m))).$$

So the set $\{x : \omega_1^x > \omega_1^{\mathrm{CK}}\}$ is $\Sigma_1^1(\mathcal{O})$. By Theorem 2.4, $\{x : \omega_1^x > \omega_1^{\mathrm{CK}}\}$ is $\Delta_1^1(\mathcal{O})$. Note that the sequence $\{\mathcal{Q}_n\}_{n \in \omega}$ is a Π_1^1-sequence, and so is an $\mathcal{O}$-recursive sequence of $\Delta_1^1(\mathcal{O})$ sets. So $\mathcal{Q} = \bigcup_{n \in \omega} \mathcal{Q}_n \cup \{x | \omega_1^x > \omega_1^{\mathrm{CK}}\}$ is a null $\Delta_1^1(\mathcal{O})$ set. Hence $\Delta_1^1(\mathcal{O})$-randomness $\Rightarrow \Pi_1^1$-randomness. By the Gandy Basis Theorem 2.1, there is a Π_1^1-random real number $x \leq_h \mathcal{O}$. Now x cannot be $\Delta_1^1(\mathcal{O})$-random. Thus the implication cannot be reversed.

Π_1^1-randomness $\Rightarrow \Pi_1^1$-ML-randomness: It is clear that Π_1^1-randomness $\subseteq \Pi_1^1$-ML-randomness. By Theorem 3.8, there exists a Π_1^1-ML-random real number $x \geq_h \mathcal{O}$. x cannot be Π_1^1-random.

Obviously Π_1^1-ML-randomness $\Rightarrow \Delta_1^1$-randomness. It follows from the Theorem 3.10 that the implication cannot be reversed.

Finally, Δ_1^1-randomness $\Leftrightarrow \Delta_1^1$-ML-randomness is simply the statement of Theorem 3.3. ∎

The reader may wonder why we do not study Σ_1^1-randomness. In fact, this is done implicitly—the following proposition says that Σ_1^1-randomness coincides with Δ_1^1-randomness.

PROPOSITION 3.13: *If A is Π_1^1 and $\mu(A) = 1$, then there is a conull Δ_1^1 set $B \subseteq A$.*

Proof. Suppose A is a Π_1^1-set for which $\mu(A) = 1$. Then, by Theorem 2.5, there is a ranked formula $\varphi(\dot{x}, y)$ so that for all $n \in \mathcal{O}_1$, $A_n \subseteq A$, where $A_n = \{x : \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \vDash \exists y^{|n|} \varphi(\dot{x}, y^{|n|})\}$). Since the set $\{x : \omega_1^x > \omega_1^{\mathrm{CK}}\}$ is null and $A_n \subseteq A_m$ for all $n <_o m$ in $\mathcal{O}_1$, by Theorem 2.5, $\mu(A) = \mu(\bigcup_{n \in \mathcal{O}_1} A_n)$. Define $R(k, n)$ if and only if $\mu(A_n) > 1 - 2^{-k}$. By Theorem 2.6, R is a Π_1^1 relation. By the Π_1^1 Uniformitarian Theorem (see [13]), there is a Π_1^1 function $f : \omega \to \omega$ uniformizing R. Since $\mu(A) = 1$, f is a total function. So f is Δ_1^1. Hence the range S of f is Δ_1^1. Then there must a recursive ordinal α so that $|n| < \alpha$ for all $n \in S$ (otherwise, $\mathcal{O}_1$ would be Δ_1^1). Fix the notation $n \in \mathcal{O}_1$ so that $|n| = \alpha$. Define $B = A_n$. Then $\mu(B) = 1$ and $B \subseteq A$.   ∎

4. Δ_1^1-dominated reals

A real number x is of hyperimmune-free degree if every function Turing reducible to x is dominated by a recursive function. We study an analog of this notion in the setting of effective descriptive set theory:

Definition 4.1: A real number x is Δ_1^1-dominated if for all functions $f : \omega \to \omega$ with $f \leq_h x$, there is a hyperarithmetic function g so that $g(n) > f(n)$ for all n (written as $g > f$).

The following contrasts with the result that the reals of hyperimmune-free degree have measure 0 ([10]; see [15] for a short proof).

THEOREM 4.2: $\mu(\{x : x \text{ is } \Delta_1^1\text{-dominated}\}) = 1$.

Proof. We prove that for any rational number p, the measure of

$$\{x : x \text{ is } \Delta_1^1\text{-dominated}\}$$

is not less than p. We apply a fusion argument to achieve this.

First, we show that for any number e, rational r, notation $n \in \mathcal{O}$ and Δ_1^1 set A for which $p + r < \mu(A)$, there is a hyperarithmetic function f so that

$$\mu(\{x : x \in A \wedge \Phi_e^{H_n^x} \text{ is total} \implies \Phi_e^{H_n^x} < f\}) > p + r/2.$$

Since the set $\{(x, i, m) : \Phi_e^{H_n^x}(i) \downarrow \implies \Phi_e^{H_n^x}(i) < m\}$ is Δ_1^1, there is a ranked formula $\varphi(\dot{x}, i, m)$ so that $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \vDash \varphi(\dot{x}, i, m)$ if and only if $\Phi_e^{H_n^x}(i) < m$.

Since A is Δ_1^1, by Theorem 2.6, the set

$$C = \{(i, m, k) : \wedge \mu(\{x : x \in A \wedge (\Phi_e^{H_n^x}(i) \downarrow \implies \Phi_e^{H_n^x}(i) < m)\}) > \mu(A) - r/2^{k+2}\}$$

is Δ_1^1. Note that for each k, there is a number m so that $(k, m, k) \in C$. So there is a Δ_1^1 total function f so that for all k, $(k, f(k), k) \in C$. Define

$$B_k = \{x : x \in A \wedge (\Phi_e^{H_n^x}(k) \downarrow \implies \Phi_e^{H_n^x}(k) < f(k))\}.$$

Then the set $\{(k, x) : x \in B_k\}$ is Δ_1^1. Moreover, for every k, $B_k \subseteq A$ and $\mu(B_k) > \mu(A) - r/2^{k+2}$. So the set $B = \bigcap_k B_k$ is Δ_1^1 and

$$\mu(B) \geq \mu(A) - \sum_{k \geq 0} \mu(A - B_k) \geq p + r - \sum_{k \geq 0} \frac{r}{2^{k+2}} = p + r/2.$$

Moreover, for every $x \in B$, if $\Phi_e^{H_n^x}$ is total, then $\Phi_e^{H_n^x} < f$. Thus we may construct an ω-sequence of Δ_1^1 sets $\{B^{\langle e, n \rangle}\}_{e \in \omega \wedge n \in \mathcal{O}}$ so that for all $e \in \omega$ and $n \in \mathcal{O}$,

 (1) If $\langle e, n \rangle > \langle e', n' \rangle$, then $B^{\langle e, n \rangle} \subseteq B^{\langle e', n' \rangle}$;
 (2) $\mu(B^{\langle e, n \rangle}) > p$.

Define $D = \bigcap_{e \in \omega \wedge n \in \mathcal{O}} B^{\langle e, n \rangle}$. Then $D \subseteq \{x : x \text{ is } \Delta_1^1\text{-dominated}\}$ and $\mu(D) \geq p$. Moreover, each real number in D is Δ_1^1-dominated. ∎

COROLLARY 4.3: *Each Π_1^1-random real number is Δ_1^1-dominated.*

Proof. By the proof of Theorem 4.2, for each $e \in \omega$ and $n \in \mathcal{O}$, the set $A_{e,n} = \{x : \exists f \in \Delta_1^1(\Phi_e^{H_n^x} \text{ is total} \implies \Phi_e^{H_n^x} < f)\}$ has measure 1. Note that $A_{e,n}$ is Π_1^1. So, by Proposition 3.13, if x is Δ_1^1-random, then $x \notin A_{e,n}$. Now if x is Π_1^1-random, then, by Proposition 3.7, $\omega_1^{CK} = \omega_1^x$. So if $g \leq_h x$, then $g = \Phi_e^{H_n^x}$ for some $e, n \in \mathcal{O}$. Thus each Π_1^1-random is Δ_1^1-dominated. ∎

Note that Π_1^1-randomness cannot be improved to Δ_1^1-randomness in Corollary 4.3. This follows from the fact that there exists a Δ_1^1-random real number $x \geq_h \mathcal{O}$ (see [5]), and from the following proposition which implies that such a real cannot be Δ_1^1-dominated.

PROPOSITION 4.4: $\{x : x \text{ is } \Delta_1^1\text{-dominated }\} \subset \{x : \omega_1^x = \omega_1^{CK}\}$.

Proof. If $\omega_1^x > \omega_1^{CK}$, then $x \geq_h \mathcal{O}$. Since there is an $\mathcal{O}$-arithmetical enumeration of Δ_1^1 functions $\{f_n\}_{n \in \omega}$, there is a $\Delta_1^1(x)$ enumeration. Define $g(n) = f_n(n) + 1$. Then $g \leq_h x$. So x is not Δ_1^1-dominated. Thus $\{x : x \text{ is } \Delta_1^1\text{-dominated }\} \subseteq \{x : \omega_1^x = \omega_1^{CK}\}$.

To see that the relation is proper, we apply Cohen forcing developed in [18]. The forcing conditions are elements of $2^{<\omega}$. A real number is said to be generic if each Σ_1^1-sentence or its negation is forced by a finite initial segment of x. So generic reals form a comeager set. Feferman (see [2] or [18]) proved that $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$ satisfies Δ_1^1-comprehension for any generic real number x. So $\omega_1^x = \omega_1^{\mathrm{CK}}$ (see [18]). We claim that no generic real can be Δ_1^1-dominated.

Given a real number x, define $g_x(n) = m_n$ if m_n is the n-th bit of x so that $x(m_n) = 1$. So there is a recursive functional Φ such that $\Phi^x = g_x$ for all x. Hence, there is a ranked (and so Σ_1^1) formula φ defining g_x, i.e. $g_x(n) = m \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi(\dot{x}, n, m)$. For any Δ_1^1 function f, there is a ranked formula ψ_f defining f, i.e. $f(n) = m \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \psi_f(n, m)$. So if $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \forall n(f(n) > g_{\dot{x}}(n))$, then there is a finite string $p \prec x$ so that $p \Vdash \forall n(f(n) > g_{\dot{x}}(n))$. This is impossible since one can easily find a condition q stronger than p so that $q \Vdash \exists n(f(n) < g_{\dot{x}}(n))$.

Thus $\{x \,|\, x \text{ is } \Delta_1^1\text{-dominated}\} \subset \{x \,|\, \omega_1^x = \omega_1^{\mathrm{CK}}\}$. ∎

One might conjecture that, by analogy to the effective case, the Δ_1^1-dominated reals form a basis for Σ_1^1 sets. This is, however, false.

PROPOSITION 4.5: *There is a nonempty Σ_1^1 set $A \subseteq 2^\omega$ which does not contain any Δ_1^1-dominated real.*

Proof. As in the proof of Proposition 4.4, there is a recursive functional Φ so that the set $A = \{x : \forall f \in \Delta_1^1 (f \not\geq \Phi^x)\}$ is nonempty. By Theorem 2.2 (the Spector-Gandy Theorem), A is a nonempty Σ_1^1 set. ∎

5. Δ_1^1-traceable reals

Next we consider the notions analogous to being r.e. traceable and recursively traceable in first order randomness theory, both of which are studied in [22, 9] (see, for instance, [9, Section 2.2] for the formal definition). The corresponding notions are called Π_1^1-traceability and Δ_1^1-traceability, respectively. We shall show that they are in fact equivalent.

Definition 5.1:   (i) Let $h : \omega \to \omega$ be a nondecreasing unbounded function that is hyperarithmetical. A Π_1^1-trace/Δ_1^1-trace with bound h is a uniformly Π_1^1/uniformly Δ_1^1 sequence $(T_e)_{e \in \omega}$ such that $|T_e| \leq h(e)$ for each e.

(ii) $A \subseteq \omega$ is Π^1_1-traceable/Δ^1_1-traceable if there is $h \in \Delta^1_1$ such that, for each $f \leq_h A$, there is a Π^1_1-trace/Δ^1_1-trace with bound h such that, for each e, $f(e) \in T_e$.

Note that, if $(T_e)_{e \in \omega}$ is a uniformly Δ^1_1 sequence of finite sets, then there is $g \in \Delta^1_1$ such that for each e, $D_{g(e)} = T_e$ (where D_n is the n-th finite set according to some recursive ordering). Thus

$$g(e) = \mu n \, \forall u \, [u \in D_n \leftrightarrow u \in T_e].$$

In this formulation, the definition of Δ^1_1 traceability is very close to that of recursive traceability. It is not difficult to see that every Δ^1_1-traceable real number is Δ^1_1-dominated.

Also notice that the choice of a bound as a witness for traceability is immaterial:

PROPOSITION 5.2 (Terwijn and Zambella [22]): *Let A be a real number that is Δ^1_1 traceable with bound h. Then for any monotone and unbounded Δ^1_1 function h', A is Δ^1_1 traceable with bound h'. The same holds for Π^1_1 traceability.*

The class of r.e. traceable sets is strictly larger than the class of recursively traceable sets, since the former contains nonrecursive r.e. sets [9]. In contrast, we have the following equivalence:

PROPOSITION 5.3: *If x is Π^1_1-traceable, then x is Δ^1_1-traceable.*

Proof. We first claim that $\omega^x_1 = \omega^{\mathrm{CK}}_1$. Otherwise, $x \geq_h \mathcal{O}$. So it is sufficient to show that $\mathcal{O}$ is not Π^1_1-traceable. Since each Π^1_1 set is many-one reducible to $\mathcal{O}$ [16, 5.4 I], there is a uniformly $\mathcal{O}$-recursive list $(T^e)_{e \in \omega}$ of all Π^1_1-traces for the bound $h(e) = e$. Define $f \leq_h \mathcal{O}$ by

$$f(e) = \mu n \, [n \notin T^e_e],$$

then f does not have a Π^1_1 trace.

To complete the proof, given $f \leq_h x$, there is a Π^1_1 trace $(T_e)_{e \in \omega}$ such that $f(e) \in T_e$ for each e. Then there is a recursive function $h : \omega^2 \to \omega$ so that $k \in T_e$ if and only if $h(k, e) \in \mathcal{O}$. Define a $\Pi^1_1(x)$-relation $R \subseteq \omega \times \mathcal{O}$ by

$$(e, n) \in R \Leftrightarrow h(e, f(e)) \in \mathcal{O}_n,$$

where $\mathcal{O}_n = \{m \in \mathcal{O} : |m| < |n|\}$, a Δ^1_1 set. Note that for each e, there is a notation $n \in \mathcal{O}$ so that $(e, n) \in R$. By the Kreisel Uniformization Theorem,

there is a total $\Pi_1^1(x)$ (and so $\Delta_1^1(x)$) function g uniformizing R. Hence the range $S = \{n : \exists e[g(e) = n]\}$ of g is a $\Delta_1^1(x)$ set. Since $\omega_1^x = \omega_1^{\mathrm{CK}}$, there exists a notation $n_0 \in \mathcal{O}$ so that $S \subseteq \mathcal{O}_{n_0}$ (otherwise the well-founded relation "$i <_o j$" would be $\Delta_1^1(x)$). Define a set $\hat{T}_e \subseteq T_e$ as follows:

$$k \in \hat{T}_e \Leftrightarrow h(k, e) \in \mathcal{O}_{n_0}.$$

By the definition of n_0, $f(e) \in \hat{T}_e$ for all $e \in \omega$. Note that the relation $n \in \hat{T}_e$ is Δ_1^1. Hence $(\hat{T}_e)_{e \in \omega}$ is a Δ_1^1-trace for f. So f is Δ_1^1-traceable. ∎

THEOREM 5.4: *There are $2^{\aleph_0}$-many Δ_1^1-traceable reals.*

Proof. We apply Sacks forcing to show this (see [18]). The forcing conditions are perfect trees coded by Δ_1^1 reals. A real number x is Sacks generic if for each Σ_1^1 sentence φ, there is a condition T so that $x \in T$ and $T \Vdash \varphi$ or $T \Vdash \neg\varphi$. Sacks proved that the set $\{(T, n_\varphi) : \varphi \in \Sigma_1^1 \wedge T \Vdash \varphi\}$ is Π_1^1. We claim that each Sacks generic real number is Δ_1^1-traceable. Thus there are $2^{\aleph_0}$-many Δ_1^1-traceable reals.

Suppose x is a Sacks generic real. Since x has minimal hyperdegree (see [18]), $\omega_1^{\mathrm{CK}} = \omega_1^x$. So if $f \leq_h x$, then there is a number e and a notation $n \in \mathcal{O}$ so that $\Phi_e^{H_n^x} = f$. Since the set $A = \{(y, n, i, j) : \Phi_e^{H_n^y}(i) = j\}$ is Δ_1^1, there exists a ranked formula defining A. Since $\Phi_e^{H_n^x}$ is total, by the definition of Sacks genericity, there is a condition $T \Vdash$ "$\Phi_e^{H_n^{\dot{x}}}$ is total". We show that for each condition $S \subseteq T$, there is a condition $Q \subseteq S$ so that $Q \Vdash$ "$\exists f(f \in \Delta_1^1 \wedge \forall i(\Phi_e^{H_n^{\dot{x}}}(i) \in D_{f(i)} \wedge |D_{f(i)}| \leq 2^{i+1}))$". Then, by the definition of forcing, there is a Δ_1^1 function f so that for all i, $\Phi_e^{H_n^{\dot{x}}}(i) \in D_{f(i)} \wedge |D_{f(i)}| \leq 2^{i+1}$.

Since $T \Vdash$ "$\Phi_e^{H_n^{\dot{x}}}$ is total", $S \Vdash$ "$\Phi_e^{H_n^{\dot{x}}}$ is total".

CASE (1): There is a condition $R \subseteq S$ so that for all i, j_0, j_1, for all conditions $P_0, P_1 \subseteq R$, $P_0 \Vdash \Phi_e^{H_n^{\dot{x}}}(i) = j_0$ and $P_1 \Vdash \Phi_e^{H_n^{\dot{x}}}(i) = j_1$ implies $j_0 = j_1$. Then we define $f(i) = j$ if and only if there exists a condition $P \subseteq R$ so that $P \Vdash \Phi_e^{H_n^{\dot{x}}}(i) = j$. Then f is a total Π_1^1-function and hence Δ_1^1. This implies that $R \Vdash f = \Phi_e^{H_n^{\dot{x}}}$.

CASE (2): Otherwise. Define a relation $\mathcal{R}(P, \sigma, i, j_0, j_1, Q_0, Q_1)$ if and only if $i \geq |\sigma|$, $j_0 \neq j_1$, $Q_0 \cap Q_1 = \emptyset$ and $Q_k \subseteq P \wedge Q_k \Vdash \Phi_e^{H_n^{\dot{x}}}(i) = j_k$ for $k \leq 1$. Obviously $\mathcal{R}$ is a Π_1^1 relation. By Kreisel's Uniformization Theorem, there is a

partial Π_1^1 function $F : 2^\omega \times 2^{<\omega} \to (\omega)^3 \times (2^\omega)^2$ so that $\mathcal{R}(P, \sigma, i, j_0, j_1, Q_0, Q_1)$ for some i, j_0, j_1, Q_0, Q_1 if and only if $\mathcal{R}(P, \sigma, F(P, \sigma))$. Without loss of generality, we assume that if $P \Vdash \Phi_e^{H_n^{\dot{x}}}(i) = j_i$ then for all $k \leq i$, $P \Vdash \Phi_e^{H_n^{\dot{x}}}(k) = j_k$ for some j_k. We do an induction on ω. During the construction, we will define a Π_1^1 sequence of conditions $\{P_\sigma\}_{\sigma \in 2^{<\omega}}$.

   Step 0. Define $P_\emptyset = S$.
   Step $n + 1$. For each $\sigma \in 2^n$, define $P_{\sigma^\frown 0} = Q_0$, $P_{\sigma^\frown 1} = Q_1$ if $F(P_\sigma, \sigma) = (i, j_0, j_1, Q_0, Q_1)$.
   Define $G(\sigma) = P_\sigma$. Then G is a total Π_1^1 and so Δ_1^1 function.
   Note that for each σ, $G(\sigma^\frown 0) \cap G(\sigma^\frown 1) = \emptyset$ and if $\sigma \preceq \tau$ then $G(\sigma) \supseteq G(\tau)$. Define

$$Q = \bigcap_n \bigcup_{\sigma \in 2^n} G(\sigma).$$

Then Q is a Δ_1^1 perfect set.
   Define a function $g : \bigcup_{i \in \omega} i \times 2^{i+1} \to \omega$ so that $g(i, \sigma) = k$ if $\sigma \in 2^{i+1}$ and $G(\sigma) \Vdash \Phi_e^{H_n^{\dot{x}}}(i) = k$. Hence g is a total Π_1^1 and therefore Δ_1^1 function. Define $f(i) = j$ if j is the least number such that $D_j = \{g(i, \sigma) : \sigma \in 2^{i+1}\}$. Then f is a Δ_1^1 function and $|D_{f(i)}| \leq 2^{i+1}$ for all i. Since for all i, $Q \subseteq \bigcup_{\sigma \in 2^n} G(\sigma)$, it is easy to see that $Q \Vdash \Phi_e^{H_n^{\dot{x}}}(i) \in D_{f(i)}$.
   So x is Δ_1^1-traceable. ■

6. Lowness for Δ_1^1-randomness

Definition 6.1: Given a relativizable class of reals $\mathcal{C}$ (for instance, $\mathcal{C}$ is the class of random reals), a real number x is low for $\mathcal{C}$ if $\mathcal{C} = \mathcal{C}^x$.

   For a randomness notion $\mathcal{C}$, we have $\mathcal{C}^x \subseteq \mathcal{C}$, and usually one would expect $\mathcal{C}^x$ to be a proper subset of $\mathcal{C}$. Thus being low for $\mathcal{C}$ means to be computationally weak, in the sense that the extra computational power of x does not help to recognize more reals as nonrandom.

   It is shown in [5] that x is low for Π_1^1-ML-randomness if and only if x is hyperarithmetical. The main result of this section is that a real number is low for Δ_1^1-randomness if and only if it is Δ_1^1-traceable. This corresponds to the main result in [9] that a real number A is low for Schnorr randomness if and only if it is recursively traceable. That result was an extension of the theorem in [22] that A is low for Schnorr tests if and only if it is recursively traceable. The

equivalence of (i) and (ii) in the theorem below reveals this parallel phenomenon in the realm of effective descriptive set theory.

For $D \subset 2^{<\omega}$ we let $[D]^{\preceq}$ denote the open set $\bigcup\{[\sigma] | \sigma \in D\}$. We often identify an open set with the corresponding set of strings closed under extension. We let S_e be the e-th finite subset of $2^{<\omega}$ under a suitable effective enumeration. Thus S_e is a finite set of strings, and $[S_e]^{\preceq} = \bigcup_{\sigma \in S_e}[\sigma]$ is then the clopen set coded by $e \in \omega$.

THEOREM 6.2: *The following are equivalent for a real number x.*

    (i) *x is Δ_1^1-traceable (or equivalently, Π_1^1 traceable).*
    (ii) *Each $\Delta_1^1(x)$ null set is contained in a Δ_1^1 null set.*
    (iii) *x is low for Δ_1^1-randomness.*
    (iv) *Each Π_1^1-ML-random set is $\Delta_1^1(x)$-random.*

Proof. (i) $\rightarrow$ (ii): Assume that x is Δ_1^1-traceable. Let $\mathcal{S}$ be a $\Delta_1^1(x)$ null set. By Lemma 3.2 relativized to x, $\mathcal{S} \subseteq \bigcap U_n$ for a $\Delta_1^1(x)$-ML test $\{U_n\}_{n \in \omega}$ such that $\mu(U_n) = 2^{-n}$ for each n. There is a function $f \leq_h x$ such that $[S_{f(\langle n,s \rangle)}]^{\preceq} =: U_{n,s}$ satisfies $U_{n,s} \subseteq U_{n,s+1}$, $U_n = \bigcup_{s \in \omega} U_{n,s}$, and, moreover, $\mu(U_{n,s}) > 2^{-n}(1 - 2^{-s})$.

Let $T = (T_e)_{e \in \omega}$ be a Δ_1^1 trace of f. By Proposition 5.2, we may choose T such that in addition $|T_e| \leq e$ for each $e > 0$.

We now define a subtrace $\hat{T}$ of T, i.e., $\hat{T}_{\langle n,s \rangle} \subseteq T_{\langle n,s \rangle}$ for each n, s. The objective is to define open sets V_n via $\hat{T}$ (in a way to be specified) small enough to give us a Δ_1^1- null set $\mathcal{V} = \bigcap_n V_n$, yet large enough as to keep all "relevant" reals out of $T_{\langle n,s \rangle} - \hat{T}_{\langle n,s \rangle}$, so that $\bigcap_{n \in \omega} U_n \subseteq \mathcal{V}$.

Let $\hat{T}_{\langle n,s \rangle}$ be the set of $e \in T_{\langle n,s \rangle}$ such that $2^{-n}(1 - 2^{-s}) \leq \mu([S_e]^{\preceq}) \leq 2^{-n}$ and $[S_e]^{\preceq} \supseteq [S_d]^{\preceq}$ for some $d \in \hat{T}_{\langle n,s-1 \rangle}$ (where $\hat{T}_{\langle n,-1 \rangle} = \omega$). Note that $f(\langle n,s \rangle) \in \hat{T}_{\langle n,s \rangle}$. Let

$$V_n = \bigcup \left\{ [S_e]^{\preceq} | e \in \hat{T}_{\langle n,s \rangle}, \ s \in \omega \right\}.$$

Then $\mu(V_n) \leq 2^{-n}|\hat{T}_{\langle n,0 \rangle}| + \sum_{s \in \omega} 2^{-s} 2^{-n}|\hat{T}_{\langle n,s \rangle}|$. Since

$$|\hat{T}_{\langle n,s \rangle}| \leq |T_{\langle n,s \rangle}| \leq \langle n,s \rangle$$

for $\langle n,s \rangle \neq 0$, and $\langle n,s \rangle$ has only polynomial growth in n and s, it is clear that $\lim_n \sum_{s \in \omega} 2^{-s} 2^{-n}|\hat{T}_{\langle n,s \rangle}| = 0$, and hence $\lim_n \mu(V_n) = 0$. Then $\mathcal{V} = \bigcap_n V_n$ is a Δ_1^1-null set and $\bigcap U_n \subseteq \mathcal{V}$.

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are immediate.

(iv) $\Rightarrow$ (i): In [9, Lemma 4.7], it is shown that, if each ML-random set is Schnorr random relative to x, then x is r.e. traceable. With merely notational changes, the proof works in the present situation. First some preliminaries. Recall that $K(\sigma)$ denotes the Π_1^1 version of prefix free Kolmogorov complexity. For $b \in \omega - \{0\}$, let $R_b = [\{\sigma \in 2^{<\omega} : K(\sigma) \le |\sigma| - b\}]$. In [5, Theorem 3.9] it is shown that $(R_b)_{b \in \omega}$ is a universal test for Π_1^1-ML-randomness. Thus, by our hypothesis in (iv), we have $\mathcal{C} \subseteq \bigcap_b R_b$ for each $\Delta_1^1(x)$ null set $\mathcal{C}$.

For $k, l \in \omega$ define the clopen set

$$B_{k,l} = \bigcup \left\{ [\tau 1^k] : \tau \in 2^{<\omega}, |\tau| = l \right\},$$

where 1^k is a string of 1's of length k. Note that $\mu(B_{k,l}) = 2^{-k}$ for all l.

Given $\sigma \in 2^{<\omega}$ and a measurable set $C \subseteq 2^\omega$, let $\mu_\sigma(C) = \frac{\mu(C \cap [\sigma])}{\mu[\sigma]}$. For an open set W let

$$W | \sigma = \bigcup \left\{ [\tau] : \tau \in 2^{<\omega}, [\sigma \tau] \subseteq W \right\}.$$

Now to find a trace for a given function $g \le_h x$, define the $\Delta_1^1(x)$-ML test U^g by stipulating that

$$U_n^g = \bigcup_{k>n} B_{k,g(k)}.$$

Hence by assumption $\bigcap_n U_n^g \subseteq \bigcap_{b \in \omega} R_b$. Thus $V = R_3$ contains $\bigcap_n U_n^g$ and $\mu(V) < 1/4$. We may assume throughout that $g(k) \ge k$ for every k because from a trace for $g(k) + k$ one can obtain a trace for g with the same bound. By [9, Lemma 4.4], there exist σ and n such that $\mu_\sigma(U_n^g - V) = 0$ and $\mu_\sigma(V) < 1/4$. As $U_0^g \supseteq U_1^g \supseteq \cdots$, we can choose σ and n with the additional property $n \ge |\sigma|$. Hence for each $k > n$, we have $g(k) \ge k > n \ge |\sigma|$ and hence $g(k) \ge |\sigma|$.

Let $\tilde{V} = V | \sigma$, let $\tilde{g}(k) = \max\{0, g(k) - |\sigma|\}$, and

$$T_k = \left\{ l : \mu(B_{k,l} - \tilde{V}) < 2^{-(l+3)} \right\}.$$

Note that for each $l \in \omega$, if $l \ge |\sigma|$ then $B_{k,l} | \sigma = B_{k,l-|\sigma|}$. So since $g(k) \ge |\sigma|$,

$$U_n^g | \sigma = \bigcup_{k>n} B_{k,g(k)} | \sigma = \bigcup_{k>n} B_{k,g(k)-|\sigma|} = U_n^{\tilde{g}},$$

and we obtain $\mu(U_n^{\tilde{g}} - \tilde{V}) = \mu_\sigma(U_n^g - V) = 0$. Hence $\tilde{g}(k) \in T_k$ for all $k > n$.

Since $\tilde{V}$ is a Π_1^1 open set, it is evident that T is a Π_1^1 set of integers. A trace for g is obtained as follows

$$G_k = \begin{cases} \{l + |\sigma| | l \in T_k\} & \text{if } k > n; \\ \{g(k)\} & \text{if } k \le n. \end{cases}$$

We now show that G is a trace for g, i.e. for all $k \in \omega$, $g(k) \in G_k$. If $k \leq n$ then this holds by definition of G_k. Thus assume $k > n$. Then $g(k) > k > n > |\sigma|$, so $\tilde{g}(k) = g(k) - |\sigma|$ so $g(k) = \tilde{g}(k) + |\sigma|$. As $k > n$, $\tilde{g}(k) \in T_k$ and hence $g(k) \in G_k$.

Clearly G is Π_1^1; so it remains to show that $|G_k|$ is hyperarithmetically bounded, independently of g. As $|G_k| = |T_k|$ for $k > n$ and $|G_k| = 1$ for $k \leq n$, this is a consequence of Lemma 4.8 of [9], reproduced below: ∎

LEMMA 6.3 ([9]): *If $\tilde{V}$ is a measurable set with $\mu(\tilde{V}) < 1/4$, and $T_k = \{l : \mu(B_{k,l} - \tilde{V}) < 2^{-(l+3)}\}$, then for $k \geq 1$, $|T_k| < 2^k k$.*

COROLLARY 6.4: *There exists a Δ_1^1-dominated real number which is not Δ_1^1-traceable.*

Proof. By Theorem 4.2, Δ_1^1-dominated reals form a measure 1 set but, by Theorem 6.2, the set of Δ_1^1-traceable reals form a null set, being disjoint from the set of Δ_1^1-random reals. ∎

7. Concluding remarks

A real number x is said to be Π_1^1-**random cuppable**, or **random cuppable** for short, if $x \oplus y \geq_h \mathcal{O}$ for all Π_1^1-random reals y. It is known [5] that if x is low for Π_1^1-randomness then $\omega_1^{\text{CK}} = \omega_1^x$. Harrington, Nies and Slaman have obtained a further result on lowness for Π_1^1-randomness. We include a proof of this result here.

THEOREM 7.1 (with Harrington and Slaman): *A real x is low for Π_1^1-randomness if and only if x is low for Δ_1^1-randomness and not random cuppable.*

Proof. For the direction from left to right, suppose x is low for Π_1^1-randomness, that is, each Π_1^1-random real number is $\Pi_1^1(x)$-random. Since $x \not\geq_h \mathcal{O}$, the $\Pi_1^1(x)$ set $\{y : y \oplus x \geq_h \mathcal{O}\}$ is null, by relativizing Theorems 2.3 and 2.4. Thus x is not random cuppable. To see that x is low for Δ_1^1-randomness, suppose for a contradiction that y is a Δ_1^1-random real number that is not $\Delta_1^1(x)$-random. Thus there is a $\Delta_1^1(x)$-null set A containing y. By the main result in Martin-Löf [11], the null set $B = \bigcup\{C \subseteq 2^\omega : \mu(C) = 0 \wedge C \text{ is } \Delta_1^1\}$ is Π_1^1. Since $y \in A - B$, $A - B$ is a nonempty $\Sigma_1^1(x)$ set. By the Gandy Basis Theorem 2.1 relative to x, there is a real number $z \in A - B$ so that $\omega_1^{z \oplus x} = \omega_1^x = \omega_1^{\text{CK}}$. Then z is

Δ_1^1-random but not $\Delta_1^1(x)$-random, so by Corollary 3.5 and its relativization to x, z is Π_1^1-random but not $\Pi_1^1(x)$-random, a contradiction.

For the other direction, suppose x is low for Δ_1^1-randomness and not random cuppable. Then $x \not\geq_h \mathcal{O}$. Suppose z is a Π_1^1-random real. By the proof of Theorem 3.4 relative to x, the largest $\Pi_1^1(x)$ null set $\mathcal{Q}(x)$ is a union of countably many $\Delta_1^1(x)$ null sets $\mathcal{Q}_n(x)$ and the $\Pi_1^1(x)$ null set $\{y : y \oplus x \geq_h \mathcal{O}\}$. Since x is low for Δ_1^1-randomness, $z \notin \bigcup_n \mathcal{Q}_n(x)$. Since x is non-Π_1^1-random cuppable, $z \oplus x \not\geq_h \mathcal{O}$. So z is $\Pi_1^1(x)$-random. ∎

The following question remains open.

QUESTION 7.2: *Is there a real number x that is low for Π_1^1-randomness but not hyperarithmetical?*

Reimann and Slaman have shown that if x is not 1-random relative to any continuous measure, then x is hyperarithmetical. In an analogy, one can ask:

QUESTION 7.3: *Is there a characterization of the reals x that are not Π_1^1-ML-random, or the ones that are not Δ_1^1-random relative to any continuous measure?*

One may also study higher genericity theory as has been done for classical genericity theory ([23] and [20]). The third author has proved that lowness for Π_1^1-genericity is the same as being hyperarithmetical and there exists a non-hyperarithmetical real number that is low for Δ_1^1-genericity.

The results of the previous sections show that several of the key notions of randomness, demonstrably different in first order theory, coalesce into equivalent ones in effective descriptive set theory. Thus finer distinctions are revealed only at the arithmetic level. It is tempting to venture beyond Π_1^1 and Δ_1^1 and explore the landscape of definable randomness in the analytical hierarchy. However, this will lead us very quickly to statements undecidable in ZFC. Assuming **projective determinacy** (PD), Kechris [6] has proved several measure and category-theoretic results in the analytical hierarchy in parallel with results for the Π_1^1 case in [17].[1] We believe that most of the results proved in the previous sections remain valid upon replacing Π_1^1 with Π_{2n+1}^1 or Σ_{2n}^1 under PD. However, it seems that PD is not a correct tool to use for analyzing the analytical sets

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[1] Since one may apply PD to obtain some dynamic properties of Π_{2n+1}^1 and Σ_{2n}^1-sets, such as scales (see [12]).

since it provides limited recursion-theoretic information. For example, PD does not give a ramified analytical hierarchy with properties similar to what one has for Π_1^1 sets. Instead, some deep results in inner model theory are necessary for this. Inner model theory (say Q-theory [8]) has been applied by some to study descriptive set theory in order to obtain powerful characterizations of analytical sets (under large cardinal assumptions, see [8])[2]. The results are of recursion-theoretic interest, and this area is worth further investigation.

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